Planar graphs have exponentially many 3-arboricities

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Abstract

It is well-known that every planar or projective planar graph can be 3-colored so that each color class induces a forest. This bound is sharp. In this paper, we show that there are in fact exponentially many 3-colorings of this kind for any (projective) planar graph. The same result holds in the setting of 3-list-colorings.

Keywords: Planar graph, vertex-arboricity, digraph chromatic number.

1 Introduction

Motivation for this paper comes from two directions. One is related to the arboricity of undirected planar graphs, the other one to colorings of planar digraphs. Let us recall that a partition of vertices of a graph G into classes $V_1 \cup \cdots \cup V_k$ is an *arboreal partition* if each V_i ($1 \le i \le k$) induces a forest in G. A function $f\colon V(G) \to \{1,\dots,k\}$ is called an *arboreal k-coloring* if $V_i = f^{-1}(i)$, $i = 1,\dots,k$, form an arboreal partition. The *vertex-arboricity* $a(G)$ of the graph G is the minimum k such that G admits an arboreal

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k-coloring. Note that $a(G) \le \chi(G) \le 2a(G)$, where $\chi(G)$ is the chromatic number of G. Long ago, people asked if every planar graph has arboricity 2 since this would imply the Four Color Theorem. However, planar graphs of vertex-arboricty 3 have been found (see Chartrand et al. [1]).

Let D be a digraph without cycles of length ≤ 2, and let G be the underlying undirected graph of D. A function $f\colon V(D) \to \{1, \dots, k\}$ is a k-*coloring* of the digraph D if $V_i = f^{-1}(i)$ is acyclic in D for every $i = 1, \dots, k$. Here we treat the vertex set V_i *acyclic* if the induced subdigraph $D[V_i]$ contains no directed cycles (but $G[V_i]$ may contain cycles). The minimum k for which D admits a k-coloring is called the *chromatic number* of D, and is denoted by $\chi(D)$ (see Neumann-Lara [6]). Clearly,

$$\chi(D) \le a(G).$$

While planar graphs with arboricity 3 are known, no planar digraph (without cycles of length ≤ 2) with $\chi(D) > 2$ is known. In fact, the following conjecture was proposed independently by Neumann-Lara [7] and Škrekovski in [2].

Conjecture 1.1. *Every planar digraph D with no directed cycles of length at most 2 has $\chi(D) \le 2$.*

It is an easy consequence of 5-degeneracy of planar graphs that every planar digraph D without cycles of length at most 2 and its associated underlying planar graph G satisfy

$$\chi(D) \le a(G) \le 3. \tag{1}$$

The main result of this paper is a relaxation of Conjecture 1.1 and a strengthening of the above stated inequality (1). In doing so, we also extend the result from planar graphs to graphs embedded in the projective plane. In particular, we prove the following.

Theorem 1.2. *Every planar or projective planar graph of order n has at least $2^{n/9}$ arboreal 3-colorings.*

Corollary 1.3. *Every planar or projective planar digraph of order n without cycles of length at most 2 has at least $2^{n/9}$ 3-colorings.*

Let us observe that Theorem 1.2 cannot be extended to graphs embedded in the torus since $a(K_7) = 4$ and K_7 admits an embedding in the torus. However, for every orientation D of K_7, we have $\chi(D) \le 3$ (and in some cases $\chi(D) = 3$); and it follows from the main result in [3] that every orientation

of a (simple) graph embeddable in the torus satisfies $\chi(D) \leq 3$. So it is possible that Corollary 1.3 extends to the torus. Graphs on the Klein Bottle behave nicer since K_7 can not be embedded in the Klein Bottle. Škrekovski [8] and Kronk and Mitchem [4] have shown that these graphs have arboricity at most 3.

It can be shown that a graph on the torus has arboricity at most 3 unless it contains K_7 as a subgraph. This can be used to prove that for every graph G embeddable in the torus, there exists an edge $e \in E(G)$ such that $a(G - e) \leq 3$. In this vein, we conjecture the following.

Conjecture 1.4. *For every graph G embeddable in the torus, there exists an edge $e \in E(G)$ such that $G - e$ has exponentially many 3-arboreal colorings.*

The proof of Theorem 1.2 is deferred until Section 4. Actually, we shall prove an extended version in the setting of list-colorings which we define next.

Let $\mathcal{C}$ be a finite set of colors. Given a graph G, let $L : v \mapsto L(v) \subseteq \mathcal{C}$ be a *list-assignment* for G, which assigns to each vertex $v \in V(G)$ a set of colors. The set $L(v)$ is called the *list* (or the set of *admissible colors*) for v. We say G is *L-colorable* if there is an *L-coloring* of G, i.e., each vertex v is assigned a color from $L(v)$ such that every color class induces a forest in G. A *k-list-assignment* for G is a list-assignment L such that $|L(v)| = k$ for every $v \in V(G)$.

Theorem 1.5. *Let L be a 3-list-assignment for a planar or projective planar graph G of order n. Then G has at least $2^{n/9}$ L-colorings.*

Similarly, we define list colorings for digraphs, where we insist that color classes induce acyclic subdigraphs. Corollary 1.3 then extends, as a corollary to Theorem 1.5 to the list coloring setting as well.

2 Unavoidable configurations

We define a *configuration* as a plane graph C together with a function $\delta\colon V(C) \to \mathbb{N}$ such that $\delta(v) \geq deg_C(v)$ for every $v \in V(C)$. A plane graph G *contains* the configuration (C, δ) if there is an injective mapping $h\colon V(C) \to V(G)$ such that the following statements hold:

(i) For every edge $ab \in E(C)$, $h(a)h(b)$ is an edge of G.

(ii) For every facial walk $a_1 \dots a_k$ in C, except for the unbounded face, the image $h(a_1) \dots h(a_k)$ is a facial walk in G.

(iii) For every $a \in V(C)$, the degree of $h(a)$ in G is equal to $\delta(a)$.

If v is a vertex of degree k in G, then we call it a *k-vertex*, and a vertex of degree at least k (at most k) will also be referred to as a *k^+-vertex* (*k^--vertex*). A neighbor of v whose degree is k is a *k-neighbor* (similarly k^+- and *k^--neighbor*).

The goal of this section is to prove the following theorem.

Theorem 2.1. *Every planar or projective planar triangulation contains one of the configurations listed in Fig. 1.*

Proof. The proof uses the discharging method. Assume, for a contradiction, that there is a (projective) planar triangulation G that contains none of the configurations shown in Fig. 1. We shall refer to these configurations as $Q_1, Q_2, \ldots, Q_{23}$.

Let G be a counterexample of minimum order. To each vertex v of G, we assign a *charge* of $c(v) = \deg(v) - 6$. A well-known consequence of Euler's formula is that the total charge is always negative, $\sum_{v \in V(G)} c(v) = -12$ in the plane and $\sum_{v \in V(G)} c(v) = -6$ in the projective plane, see [5]. We are going to apply the following *discharging rules*:

R1: A 7-vertex sends charge of $1/3$ to each adjacent 5-vertex.

R2: A 7-vertex sends charge of $1/2$ to each adjacent 4-vertex.

R3: An 8^+-vertex sends charge of $1/2$ to each adjacent 5-vertex.

R4: An 8^+-vertex sends charge of $2/3$ to each adjacent 4-vertex whose neighbors have degrees 8^+, 8^+, 8^+, 6.

R5: An 8^+-vertex sends charge of $3/4$ to each adjacent 4-vertex whose neighbors have degrees 8^+, 8^+, 7, 6.

R6: An 8^+-vertex sends charge of $1/2$ to each adjacent 4-vertex whose neighbors have degrees $8^+, 7^+, 7^+, 7^+$.

R7: An 8^+-vertex sends charge of 1 to each adjacent 4-vertex whose neighbors have degrees $8^+, 8^+, 6, 6$ or $8^+, 7, 7, 6$.

R8: An 8^+-vertex sends charge of $3/2$ to each adjacent 4-vertex whose neighbors have degrees $8^+, 7, 6, 6$.

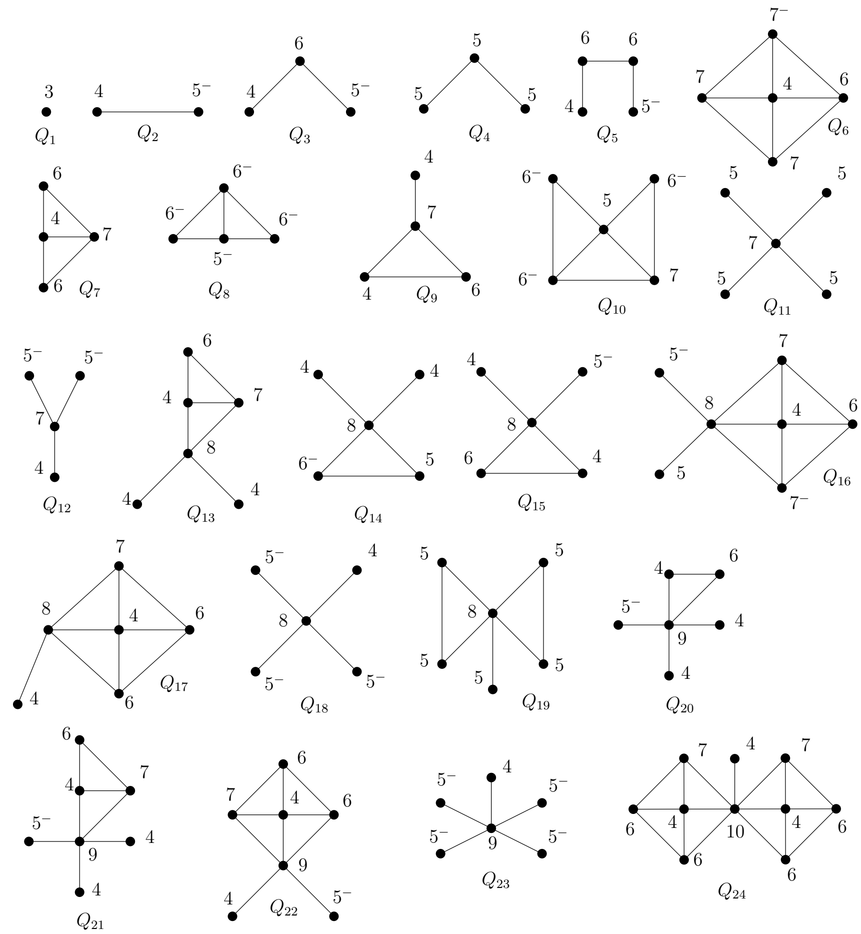

Figure 1: Unavoidable configurations. The listed numbers refer to the degree function δ, and the notation d^- at a vertex v means all such configurations where the value $\delta(v)$ is either d or $d-1$.

Let $c^*(v)$ be the *final charge* obtained after applying rules R1–R8 to all vertices in G. We will show that every vertex has non-negative final charge. This will yield a contradiction since the initial total charge of -12 (or -6 in the projective plane) must be preserved.

We say that a 4-vertex is *bad* if its neighbors have degrees $8^+, 7, 6, 6$, i.e., the rule R8 applies to it and its 8^+-neighbor. Let us observe that the clockwise order of degrees of the neighbors of a bad vertex is $8^+, 7, 6, 6$ (or $8^+, 6, 6, 7$) since Q_7 is excluded.

First, note that G has no 3^--vertices since the configuration Q_1 is ex-

cluded and since a triangulation cannot have 2^--vertices. We will also have in mind that Q_2 is excluded, so every neighbor of a 4-vertex is a 6^+-vertex.

4-vertices: Let v be a 4-vertex. Note that v has only 6^+-neighbor. If all neighbors have degree at most 7, then they all have degree exactly 7 since Q_6, Q_7 and Q_8 are excluded. Since the vertex v has initial charge of -2, and each 7-neighbor sends a charge of $1/2$ to it, the final charge of v is 0.

Now, assume that v is adjacent to an 8^+-vertex. First, assume that the remaining three neighbors v_1, v_2, v_3 of v are all 7^--vertices. The vertices v_1, v_2, v_3 cannot all have degree 6 since Q_8 is excluded. If $\deg(v_1) = 7$ and $\deg(v_2) = \deg(v_3) = 6$, then the rules R2 and R8 imply that v receives a charge of 2, resulting in the final charge of 0. If $\deg(v_1) = \deg(v_2) = 7$ and $\deg(v_3) = 6$, then by rules R2 and R7, v again receives a charge of 2. The case where $\deg(v_1) = \deg(v_2) = \deg(v_3) = 7$ is similar through rules R2 and R6.

Next, assume that v has exactly two 8^+-neighbors v_1, v_2. If the remaining two vertices v_3, v_4 are both 7-vertices, then rules R2 and R6 imply that v receives a total charge of 2, giving it the final charge of 0. If the remaining two vertices are both 6-vertices, then rule R7 implies that v receives a total charge of 2, resulting in 0 final charge. Therefore, we may assume that $\deg(v_3) = 7$ and $\deg(v_4) = 6$. In this case, both v_1 and v_2 send a charge of $3/4$ to v by R5, and v_3 sends a charge of $1/2$, resulting in a final charge of 0 for v.

Finally, assume that v has at least three 8^+-neighbors. By rule R4 (if v has a 6-neighbor), or by rules R2 and R6 (if v has a 7-neighbor), or by rule R6 (otherwise), we see that v receives total charge of 2, so $c^*(v) = 0$.

5-vertices: Let v be a 5-vertex. Note that v is not adjacent to a 4-vertex. If all neighbors of v are 7^--vertices, then exclusion of Q_4, Q_8 and Q_{10} implies that v has at least three 7-neighbors. By R1, each such neighbor sends a charge of $1/3$ to v. Since v has initial charge of -1, its final charge is at least 0. Next, suppose that v has an 8^+-neighbor. If v has at least two 8^+-neighbors, then by rule R3, v receives a charge of $1/2$ from each of them, yielding $c^*(v) \geq 0$. Therefore, we may suppose that v has exactly one 8^+-neighbor. If v has at least two 7-neighbors, then by R1 and R3, v receives a total charge of at least $1/2 + 1/3 + 1/3 > 1$, resulting in a positive final charge for v. Finally, if v has at most one 7-neighbor, then we get the configuration Q_4, Q_8 or Q_{10}.

6-vertices: They have initial charge of 0, and by the discharging rules, they do not give or receive any charge, which implies that they have a final

charge of 0.

7-vertices: Let v be a 7-vertex, and note that v has initial charge of 1. If v has no 4-neighbors then it has at most three 5-neighbors since Q_{11} is excluded. Therefore, it sends a charge of $1/3$ to each such vertex, resulting in a non-negative final charge. Next, suppose that v has at least one 4-neighbor. Since Q_{12} is excluded, v has at most one other 5^--neighbor. Therefore, v sends a charge of at most $1/2 + 1/2 = 1$, resulting in the final charge of at least 0 for v.

8-vertices: An 8-vertex v has initial charge of $+2$. Since Q_{17} is excluded, v has at most three 4-neighbors. First, suppose that v has exactly three 4-neighbors. Let u be one of them. We claim that v sends charge of at most $2/3$ to u. Since Q_{15} and Q_{13} are excluded, we have that $N(u)\backslash\{v\}$ contains vertices of degrees either $7^+, 7^+, 7^+$ or $8^+, 8^+, 6$. In the first case, v sends charge $1/2$ to u, and in the second case charge $2/3$. Since v has no 5-neighbors (again, by exclusion of Q_{17}), $c^*(v) \geq 2 - 3 \times 2/3 = 0$.

Next, suppose that v has exactly two 4-neighbors, say v_1 and v_2. We consider two subcases. First, assume that v has a 5-neighbor. Excluding Q_2 and Q_{14}, no vertex in $N(v_1) \cap N(v)$ and $N(v_2) \cap N(v)$ has degree at most 6. If the two vertices in $N(v_1) \cap N(v)$ are both 7-vertices, then v_1 has no 6^--neighbor (Q_2 and Q_{15} being excluded). This implies that v sends charge of $1/2$ to v_1. Otherwise, the two vertices in $N(v_1) \cap N(v)$ are an 8^+ and a 7^+-vertex, respectively. This implies that by rules R4, R5 and R6, v sends charge of $3/4$, $2/3$ or $1/2$ to v_1. Therefore, in all cases, v sends no more than $3/4$ charge to v_1. An identical argument shows that v sends a charge of at most $3/4$ to v_2. Since v sends a charge of $1/2$ to a 5-vertex, we have that v sends a total charge of at most $3/4 + 3/4 + 1/2 = 2$. Secondly, assume that v has no 5-neighbors. Consider v_1. Excluding Q_7 and Q_{16}, v_1 is not a bad 4-vertex. Therefore, v sends charge of at most 1 to v_1. An identical argument shows that v sends charge of at most 1 to v_2. Therefore, the final charge of v is non-negative.

Next, suppose that v has exactly one 4-neighbor, say v_1. First, suppose that v_1 is a bad 4-vertex. Excluding Q_7 and Q_{15}, v has at most one 5-neighbor. Since v sends a charge of at most $3/2$ to v_1 and charge $1/2$ to its 5-neighbor, its final charge is at least 0. Thus, we may assume that v_1 is not a bad 4-vertex. Then v sends at most charge of 1 to v_1. Because Q_{17} is excluded, v has at most two 5-neighbors, to each of which it sends a charge of $1/2$. Therefore, v sends a total charge of at most $1 + 1/2 + 1/2 = 2$, which implies that $c^*(v) \geq 0$.

Finally, suppose that v has no 4-neighbors. Excluding Q_{18} and Q_4, v has

at most four 5-neighbors, to each of which it sends charge of $1/2$. Therefore, the final charge of v is non-negative.

9-vertices: A 9-vertex v has a charge of $+3$. Since Q_{22} is excluded, v has at most four 4-neighbors. First, suppose that v has exactly four 4-neighbors or three 4-neighbors and at least one 5-neighbor; let u be one of the 4-neighbors. We claim that v sends charge of at most $2/3$ to u. Since Q_{20} and Q_{19} are excluded, we have that $N(u)\backslash\{v\}$ contains vertices of degrees $7^+, 7^+, 7^+$ or $8^+, 8^+, 6$. In the first case, v sends charge $1/2$ to u, and in the second case charge $2/3$. Since v has only one 5-neighbor (again, by exclusion of Q_{22}), $c^*(v) \geq 3 - 4 \times 2/3 > 0$.

Next, suppose that v has exactly three 4-neighbors and no 5-neighbors. Since Q_7 and Q_{21} are excluded, none of the 4-neighbors are bad. Therefore, in this case v sends charge of at most 1 to each 4-neighbor, resulting in a non-negative final charge.

If v has exactly two 4-neighbors, we consider two subcases. For the first subcase, suppose that none of the 4-neighbors are bad. Now, v has at most two 5-neighbors since Q_{22} is excluded. This implies that v sends total charge of at most $1+1+1/2+1/2 = 3$ to its neighbors, resulting in a non-negative final charge for v. For the second subcase, assume that v has at least one bad 4-neighbor. Now, the exclusion of Q_{21} implies that v has no 5-neighbors. Thus, v sends total charge of at most $3/2+3/2 = 3$, and therefore $c^*(v) \geq 0$.

Suppose now that v has exactly one 4-neighbor. The exclusion of Q_{22} implies that v has at most three 5-neighbors, and hence it sends out a total charge of at most $3/2 + 1/2 + 1/2 + 1/2 = 3$, resulting in $c^*(v) \geq 0$. Lastly, assume that v has no 4-neighbors. Excluding Q_4 we see that v has at most six 5-neighbors. This implies that v sends a total charge of at most $6 \times 1/2 = 3$ to its neighbors, thus $c^*(v) \geq 0$.

10-vertices: A 10-vertex v has a charge of $+4$. Let $v_1, \ldots, v_{10}$ be the neighbors of v in the cyclic order around v. If v_i is a bad 4-neighbor of v and $\deg(v_{i-1}) = 7$, $\deg(v_{i+1}) = 6$, then the absence of Q_3 and Q_9 implies that $\deg(v_{i+2}) \geq 6$ and $\deg(v_{i-2}) \geq 5$. The absence of Q_5 also implies that if v_{i+3} is another bad 4-neighbor, then $\deg(v_{i+2}) = 7$, thus $\deg(v_{i+4}) = 6$ and $\deg(v_{i+5}) \geq 6$ (all indices modulo 10). By excluding Q_{23} and Q_4, we conclude that if v has two bad 4-neighbors, then it has no other 4-neighbor and has at most two 5-neighbors. This implies that $c^*(v) \geq 0$. Suppose now that v has precisely one bad 4-neighbor, say v_2. We may assume $\deg(v_1) = 7$, $\deg(v_3) = 6$ and by the arguments given above, $\deg(v_{10}) \geq 5$, $\deg(v_4) \geq 6$. Excluding Q_4, v can have at most four 5-neighbors. Thus, the only possibility that $c^*(v) < 0$ is that v has three more 4-neighbors (and the only

way to have this is that the 4-neighbors are v_5, v_7, v_9) or that v has two more 4-neighbors and two 5-neighbors (in which case 4-neighbors are v_5, v_7 and 5-neighbors are v_9, v_{10}). In each of these cases, we see, by excluding Q_3 and Q_5, that $\deg(v_4) \geq 7$, $\deg(v_6) \geq 7$ and $\deg(v_8) \geq 7$. Thus, excluding Q_9, v sends charge of at most $3/4$ to each of v_5 and v_7 and at most 1 together to both v_9 and v_{10}. Hence, $c^*(v) \geq 4 - 3/2 - 2 \times 3/4 - 1 = 0$.

Suppose now that v has no bad 4-neighbors. If v has five 4-neighbors, then they are (without loss of generality) v_1, v_3, v_5, v_7, v_9, and excluding Q_3 we see that $\deg(v_j) \geq 7$ for $j = 2, 4, 6, 8, 10$. This implies (by the argument as used above) that v sends charge of at most $3/4$ to each 4-neighbor, thus $c^*(v) \geq 4 - 5 \times 3/4 > 0$. Similarly, if v has one 5-neighbor v_1 and four 4-neighbors v_3, v_5, v_7, v_9, then we see as above that v sends charge of at most $3/4$ to each 4-neighbor, and thus $c^*(v) \geq 4 - 4 \times 3/4 - 1/2 > 0$. If v has three 4-neighbors, then the exclusion of Q_4 implies that it has at most two 5-neighbors. Similarly, if v has two 4-neighbors, then it has at most four 5-neighbors. If v has one 4-neighbor, then it has at most five 5-neighbors. If v has no 4-neighbors, it has at most six 5-neighbors. In each case, $c^*(v) \geq 0$.

11^+**-vertices:** Let v be a d-vertex, with $d \geq 11$. Let $v_1, \ldots, v_d$ be the neighbors of v in cyclic clockwise order, indices modulo d. Suppose that v_i is a bad 4-vertex. Then we may assume that $\deg(v_{i-1}) = 7$ and $\deg(v_{i+1}) = 6$ (or vice versa), since Q_7 is excluded. By noting that the fourth neighbor of v_i has degree 6, we see that $\deg(v_{i+2}) \geq 6$ (since Q_3 is excluded) and $\deg(v_{i-2}) \geq 5$ (since Q_9 is excluded). If v_i is a good 4-vertex, then its neighbors are 6^+-vertices. Now, we redistribute the charge sent from v to its neighbors so that from each bad 4-vertex v_i we give $1/2$ to v_{i-1} and $1/2$ to v_{i+1}, and from each good 4-vertex v_i we give $1/4$ to v_{i-1} and $1/4$ to v_{i+1}. We claim that after the redistribution, each neighbor of v receives from v at most $1/2$ charge in total. This is clear for 4-neighbors of v. A 5-neighbor of v is not adjacent to a 4-vertex, so it gets charge of at most $1/2$ as well. The claim is clear for each 6-neighbor of v since it is adjacent to at most one 4-vertex (Q_3 is excluded). If a 7-neighbor v_j of v satisfies $\deg(v_{j+1}) = \deg(v_{j-1}) = 4$, the exclusion of Q_9 implies that both v_{j-1} and v_{j+1} are good 4-vertices. Thus, the claim holds for 7-neighbors of v. An 8^+-neighbor of v cannot be adjacent to a bad 4-neighbor of v, and therefore it receives charge of at most $1/2$ from v after the redistribution. This implies that if $d \geq 12$, then the final charge at v is $c^*(v) \geq c(v) - \frac{1}{2}d \geq 0$.

It remains to consider the case when $d = 11$. In this case the same conclusion as above can be made if we show that either the redistributed charge at one of the vertices v_i is 0, or that there are two vertices whose re-

distributed charge is at most $1/4$. If there exists a good 4-vertex, then there exists a good 4-vertex v_i, one of whose neighbors, say v_{i-1}, gets $1/4$ total redistributed charge. This is easy to see since $d = 11$ is odd and Q_3 and Q_9 are excluded. Let $t \geq 0$ be the largest integer such that $v_i, v_{i+2}, \ldots, v_{i+2t}$ are all good 4-neighbors of v. Then it is clear that v_{i+2t+1} has total redistributed charge $1/4$ and that $v_{i-1} \neq v_{i+2t+1}$ (by parity). This shows that the total charge sent from v is at most 5, thus the final charge $c^*(v)$ is non-negative. Thus, we may assume that v has no good 4-neighbors. If v has a bad 4-neighbor v_i, then we may assume that $\deg(v_{i-1}) = 7$ and $\deg(v_{i+1}) = 6$. As mentioned above, we conclude that $\deg(v_{i+2}) \geq 6$. We are done if this vertex has 0 redistributed charge. Otherwise, v_{i+2} is adjacent to another bad 4-neighbor v_{i+3} of v. Since $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ do not correspond to the excluded configuration Q_5, we conclude that $\deg(v_{i+2}) = 7$. Now we can repeat the argument with v_{i+3} to conclude that v_{i+6}, v_{i+9} are also bad 4-vertices and $\deg(v_{i+8}) = 7$. However, since $\deg(v_{i-1}) = 7$, we conclude that v_{i+9} cannot be a bad 4-vertex and hence there is a neighbor of v with redistributed charge 0.

Thus, v has no 4-neighbors. Now the only way to send charge $1/2$ to each neighbor of v is that all neighbors of v are 5-vertices. However, in this case we have the configuration Q_4.

To summarize, we have shown that the final charge of each vertex is non-negative and this completes the proof. $\square$

3 Reducibility

This section is devoted to the reducibility part of the proof of our main result (Theorem 1.5) using the unavoidable configurations in Fig. 1. Let G be a (projective) planar graph and L a 3-list-assignment. It is sufficient to prove the theorem when G is a triangulation. Otherwise, we triangulate G and any L-coloring of the triangulation is an L-coloring of G.[1] Of course, we only consider arboreal L-colorings, and we omit the adverb "arboreal" in the sequel.

A configuration C contained in G is called *reducible* if $|C| \leq 9$ and any L-coloring of $G - V(C)$ can be extended to an L-coloring of G in at least two ways. Showing that every triangulation G contains a reducible configuration will imply that G has at least $2^{|V(G)|/9}$ arboreal L-colorings.

[1] While this argument is standard for planar graphs, it is much less clear (and only conditionally true) for the case of projective plane. The details about this case are provided in the next section.

Here we prove our main theorem by showing that each configuration from Section 5.2 is reducible. The following lemma will be used throughout this section to prove reducibility.

Lemma 3.1. *Let G be a planar graph, L a 3-list-assignment for G, and $v_1, \dots, v_k \in V(G)$. Let $G_i = G - \{v_{i+1}, \dots, v_k\}$ for $i = 0, \dots, k$ and consider the following properties:*

(1) *For every $i = 1, \dots, k$, $deg_{G_i}(v_i) \leq 5$.*

(2) *There exists an i such that $deg_{G_i}(v_i) \leq 3$.*

If (1) holds, then every arboreal L-coloring of G_0 can be extended to G. If both (1) and (2) hold, then every arboreal L-coloring of G_0 can be extended to G in at least two ways.

Proof. Let f be an L-coloring of G_0. Since v_1 has degree at most 5 in G_1, there is a color $c \in L(v_1)$ such that c appears at most once on $N_{G_1}(v_1)$. Therefore, coloring v_1 with c gives an L-coloring of G_1. Repeating this argument, we see that the L-coloring of G_0 can be extended to an L-coloring of G by consecutively L-coloring $v_1, v_2, \dots, v_k$. If (2) holds for i, then there are actually two possible colors that can be used to color v_i. Therefore, every L-coloring of G_0 can be extended to G in at least two ways. □

Lemma 3.2. *Configurations $Q_1, \dots, Q_5$, $Q_8, \dots, Q_{13}$, $Q_{15}, \dots, Q_{22}$ listed in Fig. 1 are reducible. The configuration Q'_{23} that is obtained from Q_{23} by deleting the pendant vertex with $\delta(v) = 4$ is also reducible.*

Proof. For these configurations Q_i and Q'_{23} we simply apply Lemma 3.1. The corresponding enumeration $v_1, \dots, v_k$ ($k = |V(Q_i)|$ or $k = |V(Q'_{23})|$) is shown in Figure 2. The vertex for which condition (2) of Lemma 3.1 applies is always v_1; it is shown by a larger circle. □

Lemma 3.3. *Configuration Q_6 in Fig. 1 is reducible.*

Proof. Let u be the 4-vertex and let u_1, u_2, u_3, u_4 be its neighbors in cyclic order and let C be the cycle $u_1u_2u_3u_4$. Suppose that $\deg(u_1) = \deg(u_2) = 7$, $\deg(u_3) \leq 7$ and $\deg(u_4) = 6$. Let f be an L-coloring of $G - \{u, u_1, u_2, u_3, u_4\}$. Now, consider u_2. If there are at least two ways to extend the coloring f to u_2, then we can obtain at least two different colorings for G by sequentially coloring u_1, u_3, u_4, u using Lemma 3.1. Therefore, we may assume that $L(u_2) = \{1, 2, 3\}$ and that colors 1 and 2 each appear exactly twice on

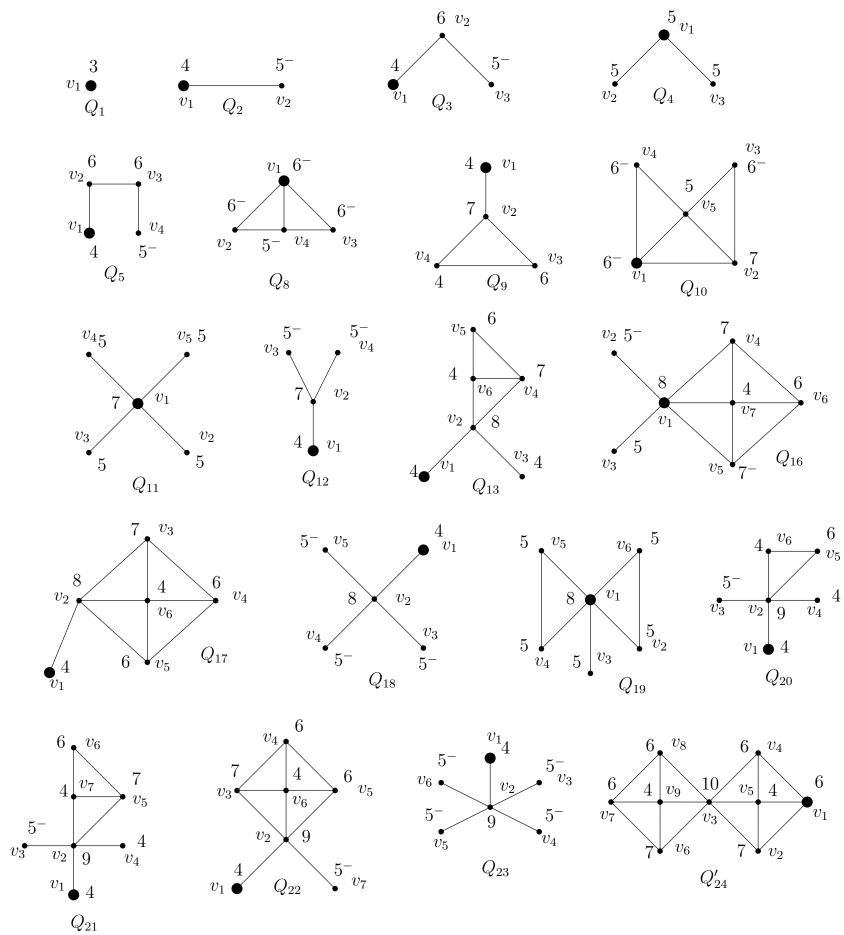

Figure 2: Lemma 3.1 applies to several configurations.

$N(u_2)$. Now, let us color u_2 with color 3. We now consider coloring u_1 and u_3. We claim that at least one of u_1 and u_3 must be forced to be colored 3. Otherwise, we color u_1 and u_3 without using color 3, then we color u_4 arbitrarily (this is possible since u is yet uncolored). Now, if $3 \in L(u)$, then we can color u with 3 since u_2 has no neighbor of color 3 and hence it is not possible to make a cycle colored 3. Moreover, there is at most one color (other than color 3) that can appear on the neighborhood of u twice. Therefore, u has another available color in its list and so there are two ways to color u. Similarly, we get two different colorings of u when $3 \notin L(u)$. This proves the claim, and we may assume that $L(u_1) = \{a, b, 3\}$, u_1 is forced to be colored 3, and that the four colored neighbors of u_1 not on C have colors

a, a, b, b. Now, we color u_3 arbitrarily with a color c. We may assume that $c \neq 3$, for otherwise we color u_4 arbitrarily and we will have two available colors for u. To complete the proof it is sufficient to show that u_4 can be colored with a color that is not c, for then we could color u with at least two different colors. If u_4 is forced to be colored c, then for every color $x \in L(u_4)$, $x \neq c$, the color x must appear at least twice on $N(u_4)$. This implies that the three colored neighbors of u_4 not on the cycle have colors $3, y, y$, for some color y and that $3, y \in L(u_4)$. But recall that u_1 and u_2 have no neighbors outside C having color 3. Therefore, coloring u_4 with color 3 gives a proper coloring of $G - u$. Now, u can be colored with at least two colors to obtain a coloring of G. □

Lemma 3.4. *Let u be a 4-vertex, and suppose u_1, u_2, u_3, u_4 are the neighbors of u in cyclic order. Suppose that* $\deg(u_1) \leq 6$, $\deg(u_2) \leq 7$ *and* $\deg(u_3) \leq 6$. *This configuration is reducible. In particular, the configuration Q_7 in Fig. 1 is reducible.*

Proof. Let f be an L-coloring of $G' = G - \{u, u_1, u_2, u_3\}$. Suppose that $f(u_4) = 3$. Now, consider u_1. Note that we can extend the coloring of G' to u_1, u_2, u_3, u (in this order) by Lemma 3.1. Suppose, for a contradiction, that f has only one extension to an L-coloring of G. Then colors of each of u_1, u_2, u_3, u are uniquely determined in each step and two colors from each vertex list are forbidden. Now, consider u_1. Since only four of its neighbors are colored and $f(u_4) = 3$, we can color u_1 with a color other than 3, say 2, and we may further assume that its colored neighbors use colors 1 and 3 twice, where $L(u_1) = \{1, 2, 3\}$. Now, consider coloring u_2. The color 2 at u_1 cannot create a monochromatic cycle containing u_2. Thus, the only way for a color of u_2 to be forced is that $L(u_2) = \{a, b, x\}$ and colors a and b each appear twice on $N(u_2) \backslash \{u_1\}$. In this case, we color u_2 with the color x. Similarly, x does not give any restriction for a color at u_3, so u_3 satisfies $L(u_3) = \{3, c, y\}$ and the three neighbors of u_3 distinct from u_4 are colored with colors $3, c, c$. Now, if u does not have two colors on $N(u)$, each appearing twice, we have two different available colors in $L(u)$. Therefore, we may assume that $\{x, y\} = \{2, 3\}$ and that $2, 3 \in L(u)$. Since $L(u_3) = \{3, c, y\}$, it follows that $y = 2$ and $x = 3$. Now we see that coloring u with color 3 does not create a monochromatic cycle, so u has two available colors: color 3 and $z \in L(u) \backslash \{2, 3\}$. □

Lemma 3.5. *The configuration Q_{14} is reducible.*

Proof. Let u be an 8-vertex and assume its neighbors (in the clockwise cyclic order) are $u_1, \ldots, u_8$ and let C be the 8-cycle $u_1 u_2 \ldots u_8 u_1$. Suppose that

$\deg(u_i) = \deg(u_j) = 4$, $\deg(u_k) \leq 5$ and $\deg(u_l) = 6$, where $i, j, k, l \in \{1, \ldots, 8\}$ and $i \neq j$. Assume that u_l and u_j are adjacent on C. We may assume that $u_l = u_{j+1}$. If $u_i = u_{j+2}$, then we can use Lemma 3.1 (with $v_1 = u_i, v_2 = u, v_3 = u_{j+1}, v_4 = u_j, v_5 = u_k$), where property (2) applies for v_1.

Therefore, we may assume that $u_i \neq u_{j+2}$. Let $L(u) = \{1, 2, 3\}$ and consider an L-coloring f of $G - \{u, u_i, u_j, u_k, u_l\}$. Without loss of generality, we may assume that colors 1 and 2 each appear exactly twice on $N(u)$ in the coloring f. Otherwise, there are two ways to extend the coloring f of $G - \{u, u_i, u_j, u_k, u_l\}$ to a coloring of $G - \{u_i, u_j, u_k, u_l\}$, and applying Lemma 3.1 we can extend each of these to a coloring of G. Therefore, color 3 does not appear in the neighborhood of u in the coloring f. We color u with color 3 to obtain a coloring g of $G - \{u_i, u_j, u_k, u_l\}$. Now, consider the 6-vertex u_{j+1}. Since u_{j+1} has at most five colored neighbors so far, we have at least one available color for it from its list. If $3 \notin L(u_{j+1})$ we color u_{j+1} arbitrarily with an available color. If $3 \in L(u_{j+1})$, we color u_{j+1} with 3 if color 3 does not appear on $N(u_{j+1}) \backslash \{u\}$. If color 3 appears on $N(u_{j+1}) \backslash \{u\}$, we color u_{j+1} with any other available color from its list except 3 (this is possible since the remaining three colored neighbors of u_{j+1} can forbid only one additional color from $L(u_{j+1})$). Now, consider u_i. We know that $u_i \neq u_{j+2}$. First, assume that $3 \notin L(u_i)$. Since u_i has only three colored neighbors and u is colored 3, there are at least two available colors in $L(u_i)$ that can be used to color u_i. Each coloring then can be extended to a coloring of G by Lemma 3.1. Therefore, we may assume that $3 \in L(u_i)$. Recall that no neighbor of u, except possibly u_{j+1}, is colored 3, and if so, then u_{j+1} has no neighbor besides u of color 3. Therefore, u_i can be colored with color 3 without creating a monochromatic cycle of color 3. Consequently, the four colored neighbors of u_i can forbid at most one color from $L(u_i)$, which implies that we can color u_i with two different colors. Now, applying Lemma 3.1 to $G - \{u_k, u_j\}$, we see that each of these two colorings can be extended to a coloring of G. $\square$

4 Proof of the main theorem

It is easy to see that every plane graph is a spanning subgraph of a triangulation; we can always add edges joining distinct nonadjacent vertices until we obtain a triangulation. However, graphs in the projective plane no longer satisfy this property. The following extension will be sufficient for our purpose.

Proposition 4.1. *Let G be a graph embeddable in the projective plane. Then one of the following holds:*

(a) *G is a spanning subgraph of a triangulation of the plane or the projective plane.*

(b) *G contains vertices u, v of degree at most 3 such that the graph $G-u-v$ is planar.*

(c) *G contains adjacent vertices u, v of degree at most 4 such that the graph $G-u-v$ is planar.*

Proof. If G is a planar graph, then we have (a); so we may assume that G is not planar. The proof proceeds by induction on the number $k = 3|V(G)| - |E(G)| - 3$. If $k = 0$, then G triangulates the projective plane (cf. [5, Proposition 4.4.4]), and we have (a). If G is not 2-connected, then we can add an edge joining two vertices in distinct blocks of G and keep the embeddability in the projective plane, and we win by induction. Thus we may assume that G is 2-connected and non-planar. This assures that facial walks of every embedding of G are cycles of G (cf. [5, Proposition 5.5.11]). If G is not a triangulation, then there is a facial cycle $C = v_1 v_2 \dots v_r v_1$, where $r \geq 4$. If two vertices of C are nonadjacent in G, we can add the edge joining them and win by induction. Thus, the subgraph K of G induced on $V(C)$ is the complete graph of order r. Since this subgraph has a facial walk of length $r > 3$, we conclude that $r \in \{4, 5\}$ and the induced embedding of K is as shown in Figure 3.

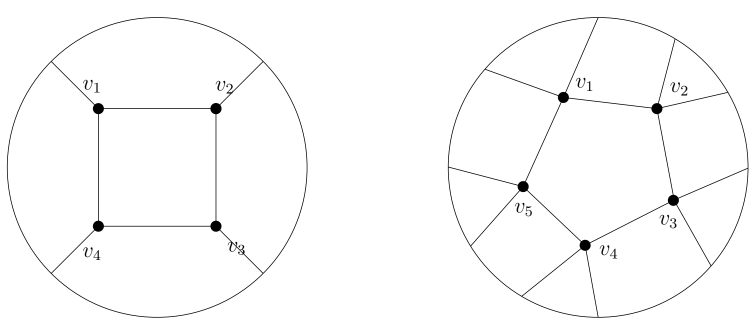

Figure 3: K_4 and K_5 embedded in the projective plane

Let us consider the vertex v_1 and the edges v_1v_3 and v_2v_4 (if $r = 4$), and v_1v_3, v_1v_4 and v_2v_5 (if $r = 5$). These edges are embedded as shown in Figure 3. Suppose that v_1 has two neighbors $a, b \notin V(C)$ such that the cyclic order

around v_1 is $v_1v_4, v_1a, v_1v_3, v_1b$ when $r = 4$ and $v_1v_5, v_1a, v_1v_s, v_1b$ (where $s = 3$ or $s = 4$) when $r = 5$. Then we can re-embed the edge v_1v_3 (if $r = 4$) or re-embed the edges v_1v_3 and v_1v_4 (if $r = 5$) into the face bounded by C and then add an edge joining two nonadjacent neighbors of v_1. Again, we are done by applying the induction hypothesis.

Thus we may assume henceforth that all neighbors of each vertex v_i that are not on C are contained in a single face of K. If a face F of K contains at least one vertex that is not on C, then each vertex of C on the boundary of F has a neighbor inside F. If not, we would be able to add an edge and would be done by applying induction. Since any two faces of K have a vertex in common, the aforementioned property implies that at most one face of K contains any vertices of G. If $r = 5$, this implies that (c) is satisfied. Thus $r = 4$ and since G is non-planar, there is a face F of K that contains vertices of G in its interior. We may assume that F contains the edge v_1v_2 on its boundary. Now, if we re-embed the edge v_1v_2 into the face of K distinct from F and C, we obtain a new face containing the former face bounded by C that is of length at least 5. Thus we get into one of the above cases, and we are done. □

Proof of Theorem 1.5. The proof is by induction on the number of vertices, $n = |G|$. Let L be a 3-list-assignment for G. Let us first suppose that G is a triangulation. By Theorem 2.1 and Lemmas 3.2–3.5, G contains a reducible configuration C on $k \leq 9$ vertices. By the induction hypothesis, $G - V(C)$ has at least $2^{(n-k)/9}$ arboreal L-colorings. Since C is reducible, each of these colorings extends to G in at least two ways, giving at least $2 \times 2^{(n-k)/9} \geq 2^{n/9}$ arboreal L-colorings in total.

If G is a spanning subgraph of a triangulation, we apply the above to the triangulation containing G. Otherwise, Proposition 4.1 shows that G contains vertices u, v of low degree such that $G-u-v$ is a spanning subgraph of a triangulation G'. By the induction hypothesis, G' has at least $2^{(n-2)/9}$ L-colorings. By properties (b) and (c) of the proposition, each of them can be extended to G in at least two ways by applying Lemma 3.1, and we conclude as before. □

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